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## Composite Propellant Technology Research: Mechanical Property Characterization

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## **1. Introduction**

This report has two sections that address aspects of the research completed to date. The first section presents proof for the existence of a single Poisson's ratio function in isotropic linear viscoelastic materials. Besides the proof the first section presents an in depth discussion of three dimensional viscoelastic material properties and their relationships for linear isotropic and orthotropic viscoelastic materials. The second section presents a discussion of the alternate invariant definition as used by Abaqus and as relates to the form used by Dr. S. Peng.

NASA's support and encouragement of this research is greatly appreciated.

## 2. Poisson's Ratio Function in Linear Viscoelasticity

### 2.1. Elements of Three Dimensional Elasticity

For completeness the discussion must begin at the most fundamental level. To be consistent with the development of linear viscoelastic theory it is first necessary to assume that displacements are small. Therefore, in the consideration of equilibrium of an infinitesimal element it is sufficient to treat the equilibrium of the element in the undeformed configuration. From the balance of angular momentum for the infinitesimal element we find that the stress tensor must be symmetric, i.e.  $\sigma_{ij} = \sigma_{ji}$ . Thus, there are only six unique components of the stress tensor at any point of a body.

The next consideration that must be addressed is the measure of the deformation. Infinitesimal strain is defined by the expression:

$$\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (1)$$

where  $\varepsilon_{ij}$  is the infinitesimal strain tensor and  $u_i$  is the displacement in the  $x_i$  coordinate direction. Following the fundamentals of calculus,  $\varepsilon_{ij} = \varepsilon_{ji}$ ; the strain tensor is also symmetric with six independent components.

Following the approach developed in the theory of elasticity the stress is related to the strain by the constitutive relationship. In elasticity this relationship is:

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl}, \quad (2)$$

where  $C_{ijkl}$  is the stiffness tensor and summation over the repeated indices is implicit as is common in tensor notation. The stiffness tensor,  $C_{ijkl}$ , is a tensor of rank four and therefore represents 81 individual constants. The inverse or compliance form of the constitutive equation is:

$$\varepsilon_{kl} = S_{klmn} \sigma_{mn}, \quad (3)$$

where  $S_{klmn}$  is the compliance tensor. As with the stiffness tensor, the compliance tensor is a tensor of rank four with 81 individual components. The compliance tensor is the inverse of the stiffness tensor; therefore:

$$C_{ijkl} S_{klmn} = \delta_{im} \delta_{jn},$$

where  $\delta_{im}$  is the identity tensor. Therefore, there are a maximum of 81 independent elastic material properties. Symmetry of the stress tensor and symmetry of the strain tensor leads to the requirements that:  $C_{ijkl} = C_{jikl} = C_{ijlk}$  and  $S_{klmn} = S_{lkmn} = S_{klnm}$ , which reduces the number of independent components to 36. Writing the stored elastic strain energy,  $U$ , as:

$$U = \frac{1}{2} \varepsilon_{ij} \sigma_{ij} ,$$

and making the substitution for the stress tensor from equation (2) yields:

$$U = \frac{1}{2} \varepsilon_{ij} C_{ijkl} \varepsilon_{kl} . \quad (4)$$

Noting then from equation (4) that there is no loss in generality by assuming the additional symmetry of the stiffness tensor such that:  $C_{ijkl} = C_{klij}$ . Therefore, the stiffness tensor (and hence the compliance tensor) has 21 independent elastic material constants.

The constitutive relations given in equations (2) and (3) with the symmetry conditions discussed are general relations with no assumptions on symmetries of material behavior. A material in which there are no symmetries of behavior is defined as an anisotropic material. Introduction of various levels of material symmetries leads to further simplifications of the constitutive relation and reductions in the number of independent material constants. Orthotropic behavior or orthorhombic symmetry leads to nine independent elastic material constants. Isotropic behavior has two independent constants.

## **2.2. Elements of One Dimensional Linear Viscoelasticity**

The foregoing discussion is primarily related to elastic behavior. For the analysis of viscoelastic behavior it is necessary to return to equation (2) and recast the relationship in that appropriate to viscoelastic behavior. A number of forms for the fundamental relation between stress and strain may be defined for viscoelastic behavior. The form assumed here is that of the historical convolution integral, i.e.:

$$\gamma(t) = \int_{-\infty}^t \frac{\partial \sigma(\tau)}{\partial \tau} J(t-\tau) d\tau, \quad (5)$$

or

$$\sigma(t) = \int_{-\infty}^t \frac{\partial \gamma(\tau)}{\partial \tau} G(t-\tau) d\tau, \quad (6)$$

where  $\gamma(\tau)$  is the strain at the time  $\tau$ ,  $\sigma(\tau)$  is the stress at the time  $\tau$ ,  $J(\tau)$  is the creep compliance function,  $G(\tau)$  is the stress relaxation modulus,  $t$  is the current time, and  $\tau$  is some time on the interval from  $-\infty$  to the current time,  $t$ . If the stress or strain histories are known to be discontinuous at some point in time it is possible to arbitrarily define that time to be the reference time, i.e.  $\tau = 0$ . Further, if either the stress or strain history is

defined to be zero for all times prior to the time  $\tau = 0$ , the historical integrals above reduce to the form:

$$\gamma(t) = \sigma(0)J(t) + \int_0^t \frac{\partial \sigma(\tau)}{\partial \tau} J(t-\tau) d\tau, \quad (7)$$

and

$$\sigma(t) = \gamma(0)G(t) + \int_0^t \frac{\partial \gamma(\tau)}{\partial \tau} G(t-\tau) d\tau. \quad (8)$$

At this point it is useful to introduce the notation of  $*$  to indicate the convolution operation. Therefore, equations (7) and (8) may be written as:

$$\gamma = J * d\sigma \quad (9)$$

and

$$\sigma = G * d\gamma, \quad (10)$$

respectively. Note that the order is  $A * dB$  where the "d" is associated with the time derivative of the function B.

Without proof the following identities are introduced for the convolution operator. The proofs are easily obtained.

$$\text{Commutativity:} \quad Q * dP = P * dQ \quad (I)$$

$$\text{Distributivity:} \quad (Q_1 + Q_2) * dP = Q_1 * dP + Q_2 * dP, \quad (II)$$

$$Q * d(P_1 + P_2) = Q * dP_1 + Q * dP_2, \quad (III)$$

$$\text{Associativity:} \quad (Q * dP) * dR = Q * d(P * dR), \quad (IV)$$

$$\text{Identity Element:} \quad Q * d1 = 1 * dQ = Q, \quad (V)$$

$$\text{Zero Element:} \quad Q * dP = 0 \text{ then } Q(t) = 0 \text{ or } P(t) = 0, \forall t, \quad (VI)$$

$$\text{Inverse Element:} \quad Q * dQ^{-1} = 1 \quad (VII)$$

$$\text{Inversion of Products:} \quad (Q * dP)^{-1} = P^{-1} * dQ^{-1} \quad (VIII)$$

where  $\mathbf{1}$  is the Heaviside step function, i.e.  $\mathbf{1}(t) = \{0, -\infty < t < 0; 1, 0 \leq t < \infty\}$ ,  $\mathbf{0}$  is the zero function, i.e.  $\mathbf{0}(t) = 0 \forall t$ , and  $(^{-1})$  indicates the inverse convolution operation. As a consequence of identity (I) the stress strain relations in equations (9) and (10) can be rewritten as:

$$\gamma = \sigma * dJ \quad (11)$$

and

$$\sigma = \gamma * dG. \quad (12)$$

These relations with the identities are used in the development of the three dimensional constitutive equations for linear viscoelastic materials.

### **2.3. Elements of Three Dimensional Linear Viscoelasticity**

The equations presented thus far for the analysis of viscoelastic response of a material have been presented in a one-dimensional form. For three dimensional viscoelastic analysis it is necessary that these equations be generalized to three dimensional behavior. Therefore, generalizing equation (2) for viscoelastic behavior yields:

$$\sigma_{ij} = C_{ijkl} * d\varepsilon_{kl}, \quad (13)$$

note that in this equation the components are all time dependent functions. Further recognize that, as a result of the convolution identities (II) and (III), this equation represents the summation of nine individual convolution integrals. The compliance form of the viscoelastic constitutive relation is:

$$\varepsilon_{kl} = S_{klmn} * d\sigma_{mn}. \quad (14)$$

Following the discussion presented for the elasticity development, the tensor functions  $C_{ijkl}(t)$  and  $S_{klmn}(t)$  must maintain the same symmetries relative to the index pairs  $i,j$ ,  $k,l$ , and  $m,n$ , therefore,  $C_{ijkl}(t) = C_{jikl}(t) = C_{ijlk}(t)$  and  $S_{klmn}(t) = S_{lkmn}(t) = S_{klnm}(t)$ ,  $\forall t$ . Then, analogous to elasticity, there are only 36 independent material property functions. For a viscoelastic material the internal energy can be written as:

$$U = \frac{1}{2} \varepsilon_{ij} * d\sigma_{ij},$$

then from (13)

$$U = \frac{1}{2} \varepsilon_{ij} * d \left\{ C_{ijkl} * d\varepsilon_{kl} \right\}.$$

The analogy with elasticity is almost complete; by use of convolution identities (I) and (IV) the order of the strain tensors may be exchanged in the above expression; hence, there is

no loss in generality as a result of the assumption that  $C_{ijkl}(t) = C_{klij}(t)$  and  $S_{klmn}(t) = S_{mnkl}(t)$ ,  $\forall t$ .

To complete the analogy with elasticity it is necessary to determine the relationship between  $C_{ijkl}(t)$  and  $S_{klmn}(t)$ . This relationship is found by substitution of (13) into (14). This yields:

$$\sigma_{ij} = C_{ijkl} * d \left\{ S_{klmn} * d\sigma_{mn} \right\},$$

and from convolution identities (I) and (IV) this becomes:

$$\begin{aligned} \sigma_{ij} &= \sigma_{mn} * d\delta_{im}\delta_{jn} = C_{ijkl} * d \left\{ \sigma_{mn} * dS_{klmn} \right\} = \left\{ C_{ijkl} * d\sigma_{mn} \right\} * dS_{klmn} \\ &= \left\{ \sigma_{mn} * dC_{ijkl} \right\} * dS_{klmn} = \sigma_{mn} * d \left\{ C_{ijkl} * dS_{klmn} \right\}. \end{aligned}$$

Therefore:

$$0 = \sigma_{mn} * d \left\{ \delta_{im}\delta_{jn} - C_{ijkl} * dS_{klmn} \right\}.$$

Finally, from convolution identity (VI) for any general stress history that satisfies the conditions discussed, i.e. zero to time  $t = 0$ , we find:

$$C_{ijkl} * dS_{klmn} = \delta_{im}\delta_{jn}, \quad (15)$$

which is similar to the result obtained for elasticity, however, not nearly as simple.

Symmetries of response that lead to the reduction of the number of independent material properties are by definition time independent. Therefore, the results obtained for elastic materials are directly applicable to viscoelastic materials. Hence, an orthotropic viscoelastic material has nine independent material property functions and an isotropic viscoelastic material has two independent material property functions.

For the investigation of the Poisson effect in viscoelastic materials it will be assumed that the material in question is at most orthotropic; later discussions will be restricted to isotropic materials. To simplify the discussion the compact notation introduced by Tsai for composite materials will be used. Compact notation is defined as follows:



$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} \text{ and } \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{Bmatrix}. \quad (16)$$

Then for an orthotropic material in which the axes of symmetry or principal material directions are aligned with the coordinate directions, i.e. a specially orthotropic material, the compact notation for the stiffness and compliance is:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ C_{1122} & C_{2222} & C_{2233} & 0 & 0 & 0 \\ C_{1133} & C_{2233} & C_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{4444} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{5555} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{6666} \end{bmatrix}, \quad (17)$$

and

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & 0 & 0 & 0 \\ S_{1122} & S_{2222} & S_{2233} & 0 & 0 & 0 \\ S_{1133} & S_{2233} & S_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{4444} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{5555} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{6666} \end{bmatrix}. \quad (18)$$

In contracted notation the interrelation of stiffness and compliance, equation (15), becomes:

$$C_{ij} * dS_{jk} = \delta_{ik}, \quad (19)$$

where the repeated index indicates a summation from 1 to 6. Expanded in matrix form the above is:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} * d \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (20)$$

which represents twelve nontrivial equations interrelating the compliance functions and stiffness functions. Recognizing that the matrices may be partitioned into four 3X3 sub-matrices, i.e.

$$\begin{bmatrix} C^{\text{Tension}} & 0 \\ 0 & C^{\text{Shear}} \end{bmatrix} * d \begin{bmatrix} S^{\text{Tension}} & 0 \\ 0 & S^{\text{Shear}} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad (21)$$

where  $C^{\text{Tension}}$  and  $C^{\text{Shear}}$  are the sub-matrices containing the tensile stiffness functions and shear stiffness functions, respectively,  $S^{\text{Tension}}$  and  $S^{\text{Shear}}$  are the corresponding compliance sub-matrices and  $I$  is the identity matrix. Written in this form it is clear that the off diagonal sub-matrices make no contribution. Therefore, the relations may be reduced to two sets of matrix equations, which are:

$$\begin{bmatrix} C^{\text{Tension}} \end{bmatrix} * d \begin{bmatrix} S^{\text{Tension}} \end{bmatrix} = \begin{bmatrix} I \end{bmatrix} \quad (22)$$

and

$$\begin{bmatrix} C^{\text{Shear}} \end{bmatrix} * d \begin{bmatrix} S^{\text{Shear}} \end{bmatrix} = \begin{bmatrix} I \end{bmatrix}. \quad (23)$$

Noting that  $C^{\text{Shear}}$  and  $S^{\text{Shear}}$  are diagonal matrices, equation (23) reduces to three independent equations, which are:

$$1 = C_{44} * dS_{44}, \quad (24)$$

$$1 = C_{55} * dS_{55}, \quad (25)$$

$$1 = C_{66} * dS_{66}. \quad (26)$$

Therefore, for a specially orthotropic viscoelastic material the shear stiffness functions and shear compliance functions are inverse convolution functions of one another. However, the same comment does not hold for the tensile terms.

#### **2.4. Engineering Properties in Linear Viscoelasticity**

If one defines the shear stress relaxation modulus functions such that:

$$\sigma_4 = C_{44} * d\varepsilon_4 = G_{23} * d\gamma_4, \quad (27)$$

$$\sigma_5 = C_{55} * d\varepsilon_5 = G_{13} * d\gamma_5, \quad (28)$$

$$\sigma_6 = C_{66} * d\varepsilon_6 = G_{12} * d\gamma_6, \quad (29)$$

where  $G_{ij}$  are the shear stress relaxation modulus functions in the  $i$ - $j$  plane and  $\gamma_k$  is the engineering shear strain in the associated plane, the relationship between the stiffnesses and moduli is a one to one relationship. Similarly, by defining the shear creep compliance functions such that

$$\varepsilon_4 = S_{44} * d\sigma_4 = J_{23} * d\sigma_4, \quad (30)$$

$$\varepsilon_5 = S_{55} * d\sigma_5 = J_{13} * d\sigma_5, \quad (31)$$

$$\varepsilon_6 = S_{66} * d\sigma_6 = J_{12} * d\sigma_6, \quad (32)$$

an analogous one to one relationship exists between the two sets of compliance functions. Then as a consequence of equations (24) - (26) and the above equations (equations (27) - (32)), the engineering creep compliance functions and stress relaxation functions for shear are also inverse convolution functions, i.e.:

$$1 = J_{12} * dG_{12}, \quad (33)$$

$$1 = J_{13} * dG_{13}, \quad (34)$$

$$1 = J_{23} * dG_{23}. \quad (35)$$

This result is in full agreement with the one dimensional approach to viscoelasticity and the results stated by Staverman and Schwarzl<sup>1</sup> for shear behavior.

Before continuing with the investigation of Poisson's effect in a viscoelastic material a discussion of the interrelation of the stiffness and compliance functions is appropriate. To begin the discussion it is first important to note that terms  $C_{ii}$  in equation (22) are not the tensile stress relaxation modulus functions, they are the tensile stiffness functions. They are dependent on the tensile (engineering) stress relaxation modulus functions as well as other functions which describe the Poisson effect of the material. The compliance terms  $S_{ii}$  are directly the tensile engineering creep compliance functions.

Therefore, by defining that  $F(t)$  is a tensile creep compliance function, after the notation of Staverman and Schwarzl, the creep compliance functions and the tensile components of the compliance tensor are:

$$S_{ii}(t) = F_i(t), \quad (36)$$

where  $F_i(t)$  is the engineering creep compliance function in the  $i$ th principal material direction.

For clarity, consider momentarily a orthotropic elastic material. In an elastic  $S_{ii} = 1/E_i$ , where  $E_i$  is the Young's modulus, or engineering elastic modulus, of the material in the  $i$ th principal material direction, and  $1/E_i$  is called the compliance of the material. Continuing along this line of thought, the off-diagonal terms of the elastic compliance,  $S_{ij}$ , are related to the compliances,  $1/E_i$ , and the Poisson's ratios,  $\nu_{ij}$ , such that  $S_{ij} = -\nu_{ij}/E_i$ . For an orthotropic material the Poisson's ratios are defined as the ratio of the strain in the transverse direction,  $j$ , relative to the strain in the loading directions,  $i$ , i.e.  $\nu_{ij} = -\varepsilon_j/\varepsilon_i$ .

The definition of the Poisson's ratio functions for an orthotropic viscoelastic material are analogous to those for an orthotropic elastic material. After the manner of Staverman and Schwarzl, it is:

$$\varepsilon_j(t) = - \int_{-\infty}^t \nu_{ij}(t-\tau) \frac{d\varepsilon_i(\tau)}{d\tau} d\tau, \quad (37)$$

where  $\nu_{ij}(t-\tau)$  is a viscoelastic Poisson's ratio function for strains in the  $j$ th direction due to strains in the  $i$ th direction. Note that there is no distinction between a Poisson's ratio function that would be associated with a creep test or relaxation test. A proof is provided later demonstrating that there is no difference in the Poisson's ratio functions that result from a creep compliance test or a stress relaxation test. Following the initial discussions relative to the loading history and the nature of the convolution integral in equation (37) the definition may be rewritten as:

$$\varepsilon_j = - \nu_{ij} * d\varepsilon_i = - \varepsilon_i * d\nu_{ij}. \quad (38)$$

For the time being it will be assumed that there is no difference between Poisson's ratio functions resulting from stress relaxation tests and creep compliance tests.

The relation between the off-diagonal compliance terms and the Poisson's ratio functions requires some development. Consider a uniaxial tensile creep compliance test in the first principal material direction, i.e.  $\sigma_1 = \sigma_0$ , and  $\sigma_2 = \sigma_3 = 0$ . Substitution into equation (14) yields:

$$\varepsilon_1 = S_{11} \sigma_0, \quad \varepsilon_2 = S_{12} \sigma_0, \quad \varepsilon_3 = S_{13} \sigma_0.$$

Then, substitution of the above into equation (38) produces:

$$\varepsilon_2 = S_{12} \sigma_o = -\nu_{12} * d\varepsilon_1 = -\nu_{12} * d[S_{11} \sigma_o] = -\sigma_o \nu_{12} * dS_{11}$$

and

$$\varepsilon_3 = S_{13} \sigma_o = -\nu_{13} * d\varepsilon_1 = -\nu_{13} * d[S_{13} \sigma_o] = -\sigma_o \nu_{13} * dS_{11}.$$

Using the definition of the tensile engineering creep compliance functions,  $F_i$ , the above equations yield:

$$S_{12} = -\nu_{12} * dF_1 \quad (39)$$

and

$$S_{13} = -\nu_{13} * dF_1. \quad (40)$$

Similarly, uniaxial tensile creep compliance tests in the second and third principal material directions yield, respectively:

$$S_{21} = -\nu_{21} * dF_2, \quad (41)$$

$$S_{23} = -\nu_{23} * dF_2, \quad (42)$$

$$S_{31} = -\nu_{31} * dF_3, \quad (43)$$

and

$$S_{32} = -\nu_{32} * dF_3. \quad (44)$$

As a consequence of symmetry of the compliance tensor (matrix) and the above relations the i-j and j-i Poisson's ratio functions are interrelated by the following equations:

$$\nu_{12} * dF_1 = \nu_{21} * dF_2, \quad (45)$$

$$\nu_{13} * dF_1 = \nu_{31} * dF_3, \quad (46)$$

and

$$\nu_{23} * dF_2 = \nu_{32} * dF_3. \quad (47)$$

These results are analogous to those found in the development of orthotropic elasticity.

Using matrix methods the tensile compliance tensor (matrix) may be expressed as:

$$\begin{aligned}
\begin{bmatrix} \mathbf{S}^{\text{Tension}} \end{bmatrix} &= \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} = \\
&= \begin{bmatrix} 1 & -\nu_{12} & -\nu_{13} \\ -\nu_{21} & 1 & -\nu_{23} \\ -\nu_{31} & -\nu_{32} & 1 \end{bmatrix} * \mathbf{d} \begin{bmatrix} F_1 & 0 & 0 \\ 0 & F_2 & 0 \\ 0 & 0 & F_3 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{P} \end{bmatrix} * \mathbf{d} \begin{bmatrix} \mathbf{F} \end{bmatrix}, \tag{48}
\end{aligned}$$

where  $\mathbf{P}$  is the Poisson's effect matrix and  $\mathbf{F}$  is the tensile compliance matrix. The Poisson's effect matrix is a nonsymmetric matrix while the tensile compliance matrix is a diagonal matrix.

Equation (20) gives the relationship between the orthotropic viscoelastic stiffness functions and the compliance functions and equation (22) gives the tensile or normal part of these relations. With equations (45) - (47), equation (22) can be rewritten as:

$$\begin{bmatrix} \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{C}^{\text{Tension}} \end{bmatrix} * \mathbf{d} \left\{ \begin{bmatrix} \mathbf{P} \end{bmatrix} * \mathbf{d} \begin{bmatrix} \mathbf{F} \end{bmatrix} \right\}. \tag{49}$$

By applying the identities for convolution integration the above equation may be rewritten as:

$$\begin{bmatrix} \mathbf{I} \end{bmatrix} = \left\{ \begin{bmatrix} \mathbf{C}^{\text{Tension}} \end{bmatrix} * \mathbf{d} \begin{bmatrix} \mathbf{P} \end{bmatrix} \right\} * \mathbf{d} \begin{bmatrix} \mathbf{F} \end{bmatrix}. \tag{50}$$

Note that the tensile compliance matrix,  $\mathbf{F}$ , is a diagonal matrix. If we introduce the functions  $F_i^{-1}(t)$ , such that  $F_i^{-1}(t)$  is the inverse convolution function of  $F_i(t)$ , i.e.

$F_i(t) * \mathbf{d}F_i^{-1}(t) = 1$  from convolution identity (VII), then we can write:

$$\begin{bmatrix} \mathbf{I} \end{bmatrix} * \mathbf{d} \begin{bmatrix} \mathbf{F}^{-1} \end{bmatrix} = \left\{ \left\{ \begin{bmatrix} \mathbf{C}^{\text{Tension}} \end{bmatrix} * \mathbf{d} \begin{bmatrix} \mathbf{P} \end{bmatrix} \right\} * \mathbf{d} \begin{bmatrix} \mathbf{F} \end{bmatrix} \right\} * \mathbf{d} \begin{bmatrix} \mathbf{F}^{-1} \end{bmatrix},$$

which, by application of convolution identity (II), becomes:

$$\begin{bmatrix} \mathbf{I} \end{bmatrix} * \mathbf{d} \begin{bmatrix} \mathbf{F}^{-1} \end{bmatrix} = \left\{ \begin{bmatrix} \mathbf{C}^{\text{Tension}} \end{bmatrix} * \mathbf{d} \begin{bmatrix} \mathbf{P} \end{bmatrix} \right\} * \mathbf{d} \left\{ \begin{bmatrix} \mathbf{F} \end{bmatrix} * \mathbf{d} \begin{bmatrix} \mathbf{F}^{-1} \end{bmatrix} \right\},$$

and, finally by application of convolution identity (V) and (VII), the above becomes:

$$\begin{bmatrix} \mathbf{F}^{-1} \end{bmatrix} = \begin{bmatrix} \mathbf{C}^{\text{Tension}} \end{bmatrix} * \mathbf{d} \begin{bmatrix} \mathbf{P} \end{bmatrix}. \quad (51)$$

Note that the matrix  $\mathbf{F}^{-1}$  is a diagonal matrix, while the matrices  $\mathbf{C}^{\text{Tension}}$  and  $\mathbf{P}$  are full matrices. Further, as previously mentioned,  $\mathbf{P}$  is a nonsymmetric matrix. The  $\mathbf{F}$  matrix is eliminated from the right hand side of the above equation by the simple introduction of  $\mathbf{F}^{-1}$  only because  $\mathbf{F}$  is a diagonal matrix.

At this point in the elasticity development, the elasticity form of the above equation is simply post multiplied by the inverse of the Poisson's matrix, resulting in an expression for the stiffness terms as functions of the inverse compliances and the Poisson terms. The viscoelastic development is not as simple. While it is mathematically and analytically feasible to introduce a matrix  $\mathbf{\Pi}$  such that the convolution of  $\mathbf{\Pi}$  with the Poisson's matrix,  $\mathbf{P}$ , produces the identity matrix, i.e.:

$$\begin{bmatrix} \mathbf{\Pi} \end{bmatrix} * \mathbf{d} \begin{bmatrix} \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{I} \end{bmatrix}, \quad (52)$$

it is computationally impractical. Allowing that the  $\mathbf{\Pi}$  matrix exists, by convolution of equation (51) with  $\mathbf{\Pi}$  yields:

$$\begin{bmatrix} \mathbf{F}^{-1} \end{bmatrix} * \mathbf{d} \begin{bmatrix} \mathbf{\Pi} \end{bmatrix} = \left\{ \begin{bmatrix} \mathbf{C}^{\text{Tension}} \end{bmatrix} * \mathbf{d} \begin{bmatrix} \mathbf{P} \end{bmatrix} \right\} * \mathbf{d} \begin{bmatrix} \mathbf{\Pi} \end{bmatrix}.$$

Then, after application of the convolution identities:

$$\begin{bmatrix} \mathbf{C}^{\text{Tension}} \end{bmatrix} = \begin{bmatrix} \mathbf{F}^{-1} \end{bmatrix} * \mathbf{d} \begin{bmatrix} \mathbf{\Pi} \end{bmatrix}, \quad (53)$$

in which  $\mathbf{F}^{-1}$  is a diagonal matrix of inverse convolution functions of the tensile creep compliance functions in the principal material directions and  $\mathbf{\Pi}$  is a full nonsymmetric matrix of functions related to the Poisson's ratio functions.

For the moment assume that the Poisson's ratio functions are known functions of time. To determine the elements of  $\mathbf{\Pi}$  requires the computation of multiple convolution integrals and inverse convolution functions. For proof consider the following development. Expand equation (52) in terms of the individual components;

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} \\ \Pi_{21} & \Pi_{22} & \Pi_{23} \\ \Pi_{31} & \Pi_{32} & \Pi_{33} \end{bmatrix} * \mathbf{d} \begin{bmatrix} 1 & -\nu_{12} & -\nu_{13} \\ -\nu_{21} & 1 & -\nu_{23} \\ -\nu_{31} & -\nu_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \Pi_{11}-\Pi_{12} * d\nu_{21}-\Pi_{13} * d\nu_{31} & \Pi_{12}-\Pi_{11} * d\nu_{12}-\Pi_{13} * d\nu_{32} & \Pi_{13}-\Pi_{11} * d\nu_{13}-\Pi_{12} * d\nu_{23} \\ \Pi_{21}-\Pi_{22} * d\nu_{21}-\Pi_{23} * d\nu_{31} & \Pi_{22}-\Pi_{21} * d\nu_{12}-\Pi_{23} * d\nu_{32} & \Pi_{23}-\Pi_{21} * d\nu_{13}-\Pi_{22} * d\nu_{23} \\ \Pi_{31}-\Pi_{32} * d\nu_{21}-\Pi_{33} * d\nu_{31} & \Pi_{32}-\Pi_{31} * d\nu_{12}-\Pi_{33} * d\nu_{32} & \Pi_{33}-\Pi_{31} * d\nu_{13}-\Pi_{32} * d\nu_{23} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Concentrating on evaluation of the  $\Pi_{11}$  term, expand the first row to three independent equations;

$$\Pi_{11} = 1 + \Pi_{12} * d\nu_{21} + \Pi_{13} * d\nu_{31} \quad (i)$$

$$\Pi_{12} = \Pi_{11} * d\nu_{12} + \Pi_{13} * d\nu_{32} \quad (ii)$$

$$\Pi_{13} = \Pi_{11} * d\nu_{13} + \Pi_{12} * d\nu_{23} \quad (iii)$$

Equations (ii) and (iii) by taking the convolution of the equations with  $\nu_{13}$  and  $\nu_{12}$ , respectively, thus:

$$\Pi_{12} * d\nu_{13} = (\Pi_{11} * d\nu_{12}) * d\nu_{13} + (\Pi_{13} * d\nu_{32}) * d\nu_{13}, \quad (iv)$$

$$\Pi_{13} * d\nu_{12} = (\Pi_{11} * d\nu_{13}) * d\nu_{12} + (\Pi_{12} * d\nu_{23}) * d\nu_{12}. \quad (v)$$

Then the  $\Pi_{11}$  term is eliminated by subtracting equation (v) from equation (iv). This yields:

$$\begin{aligned} \Pi_{12} * d\nu_{13} - \Pi_{13} * d\nu_{12} &= \\ &= (\Pi_{11} * d\nu_{12}) * d\nu_{13} + (\Pi_{13} * d\nu_{32}) * d\nu_{13} - (\Pi_{11} * d\nu_{13}) * d\nu_{12} - \\ &\quad (\Pi_{12} * d\nu_{23}) * d\nu_{12} \end{aligned}$$

$$\Pi_{12} * d\nu_{13} - \Pi_{13} * d\nu_{12} = (\Pi_{13} * d\nu_{32}) * d\nu_{13} - (\Pi_{12} * d\nu_{23}) * d\nu_{12} \quad (vi)$$

Then rearranging and applying the convolution identities gives:

$$\Pi_{12} * d(\nu_{13} + \nu_{23} * d\nu_{12}) = \Pi_{13} * d(\nu_{12} + \nu_{32} * d\nu_{13}). \quad (vii)$$

Introducing the convolution inverse of  $(\nu_{13} + \nu_{23} * d\nu_{12})$  it is possible to express  $\Pi_{12}$  in terms of  $\Pi_{13}$  and  $\nu_{ij}$ . This expression is:

$$\Pi_{12} = \left\{ \Pi_{13} * d \left[ \nu_{12} + \nu_{32} * d\nu_{13} \right] \right\} * d \left\{ \nu_{13} + \nu_{23} * d\nu_{12} \right\}^{-1}, \quad (viii)$$



where  $\left\{ \nu_{13} + \nu_{23} * d\nu_{12} \right\}^{-I}$  is treated as a single function of time such that:

$$\left\{ \nu_{13} + \nu_{23} * d\nu_{12} \right\} * d \left\{ \nu_{13} + \nu_{23} * d\nu_{12} \right\}^{-I} = 1. \quad (\text{ix})$$

By application of the convolution distribution property equation (viii) becomes:

$$\Pi_{12} = \Pi_{13} * d \left[ \left\{ \nu_{12} + \nu_{32} * d\nu_{13} \right\} * d \left\{ \nu_{13} + \nu_{23} * d\nu_{12} \right\}^{-I} \right]. \quad (\text{x})$$

Equating equation (ii) and equation (x) and rearranging gives:

$$\Pi_{13} * d \left[ \left\{ \nu_{12} + \nu_{32} * d\nu_{13} \right\} * d \left\{ \nu_{13} + \nu_{23} * d\nu_{12} \right\}^{-I} - \nu_{32} \right] = \Pi_{11} * d\nu_{12}. \quad (\text{xi})$$

Solution of (xi) for  $\Pi_{13}$  in terms of  $\Pi_{11}$  and  $\nu_{ij}$  requires introduction of another inverse convolution function where

$$\left[ \left\{ \nu_{12} + \nu_{32} * d\nu_{13} \right\} * d \left\{ \nu_{13} + \nu_{23} * d\nu_{12} \right\}^{-I} - \nu_{32} \right] * d \left[ \left\{ \nu_{12} + \nu_{32} * d\nu_{13} \right\} * d \left\{ \nu_{13} + \nu_{23} * d\nu_{12} \right\}^{-I} - \nu_{32} \right]^{-I} = 1, \quad (\text{xii})$$

as before this function must be treated as an independent function of time. Applying the result from equation (xii) to equation (xi) yields an expression for  $\Pi_{13}$ . It is:

$$\Pi_{13} = \Pi_{11} * d \left\{ \nu_{12} * d \left[ \left\{ \nu_{12} + \nu_{32} * d\nu_{13} \right\} * d \left\{ \nu_{13} + \nu_{23} * d\nu_{12} \right\}^{-I} - \nu_{32} \right]^{-I} \right\}. \quad (\text{xiii})$$

Then substituting the above equation into equation (x) produces:

$$\Pi_{12} = \Pi_{11} * d \left\{ \nu_{12} * d \left[ \left\{ \left[ \nu_{12} + \nu_{32} * d\nu_{13} \right] * d \left[ \nu_{13} + \nu_{23} * d\nu_{12} \right]^{-1} \right\} \right. \right. \\ \left. \left. * d \left[ \left[ \nu_{12} + \nu_{32} * d\nu_{13} \right] * d \left[ \nu_{13} + \nu_{23} * d\nu_{12} \right]^{-1} - \nu_{32} \right]^{-1} \right] \right\} \quad (xiv)$$

Finally, substituting equations (xiii) and (xiv) into equation (i) yields an expression for  $\Pi_{11}$  in terms of  $\nu_{ij}$ . This expression is:

$$\Pi_{11} = \left[ 1 - \nu_{21} * d \left\{ \nu_{12} * d \left[ \left\{ \left[ \nu_{12} + \nu_{32} * d\nu_{13} \right] * d \left[ \nu_{13} + \nu_{23} * d\nu_{12} \right]^{-1} \right\} \right. \right. \right. \\ \left. \left. * d \left[ \left[ \nu_{12} + \nu_{32} * d\nu_{13} \right] * d \left[ \nu_{13} + \nu_{23} * d\nu_{12} \right]^{-1} - \nu_{32} \right]^{-1} \right] \right\} \right. \\ \left. - \nu_{31} * d \left\{ \nu_{12} * d \left[ \left[ \nu_{12} + \nu_{32} * d\nu_{13} \right] * d \left[ \nu_{13} + \nu_{23} * d\nu_{12} \right]^{-1} - \nu_{32} \right]^{-1} \right\} \right] \quad (54)$$

Thus, it is possible to derive the elements of inverse convolution matrix. Computing the value of these elements as functions of time is practical impossibility in view of the need to compute three different convolution inverse functions from the known functions  $\nu_{ij}$ .

## 2.5. Stress Relaxation Testing of a Specially Orthotropic Viscoelastic Material

The previous development focused primarily on the creep compliance formulation of specially orthotropic viscoelastic response. By doing so it was possible to directly relate the creep compliance elements,  $S_{ij}$ , to the tensile and shear creep compliance functions,  $F_i$ , and  $G_{kl}$  and the viscoelastic Poisson's ratio functions,  $\nu_{ij}$ . Additionally, it was shown that the shear creep compliance functions are inverse convolution functions of the shear stress relaxation functions and finally that, while a similar relationship exists for the tensile functions, the relationship is a complex combination of the Poisson's functions. Because the shear stress relaxation functions are directly related to the creep compliance functions and thus may be computed based evaluation of the convolution inverse it is not necessary to specifically address shear stress relaxation testing.

Consider a uniaxial tensile stress relaxation test for a specially orthotropic material. In this test a uniaxial tensile strain is applied as a step function in time in the first principal material directions,  $x_1$ . Thus, the applied loading is:  $\varepsilon_1 = \varepsilon_0 1$ . The boundary conditions for the equilibrium equations for the specimen determine the complete loading of the specimen. These conditions are:  $\sigma_2 = \sigma_3 = 0$ . The normal strains in the second and third principal material directions are not known a priori, they must be determined from the constitutive equation. From the previous development:

$$\left\{ \sigma \right\} = \left[ C^{\text{Tension}} \right] * d \left\{ \varepsilon \right\}. \quad (55)$$

Then, for the stress relaxation test in the first principal material direction:

$$\left\{ \begin{matrix} \sigma_1 \\ 0 \\ 0 \end{matrix} \right\} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} * d \left\{ \begin{matrix} \varepsilon_0 1 \\ \varepsilon_2 \\ \varepsilon_3 \end{matrix} \right\}. \quad (56)$$

To produce useful results it is necessary to rearrange the equations such that the knowns are grouped together on the right hand side of the expression. When this is done we find:

$$\begin{aligned} \frac{\sigma_1}{\varepsilon_0} = & C_{11} + \left\{ C_{23} * dC_{23} - C_{22} * dC_{33} \right\}^{-1} \\ & * d \left\{ C_{33} * d \left( C_{12} * dC_{12} \right) + C_{22} * d \left( C_{13} * dC_{13} \right) - 2C_{12} * d \left( C_{13} * dC_{23} \right) \right\} \end{aligned} \quad (57)$$

Thus, the stress measured in the stress relaxation test is related to the tensile stiffness in the first material direction,  $C_{11}$ , as well as other stiffness terms. In addition, it is important to note that  $C_{11}$  is not the relaxation modulus,  $E_1(t)$ , in the first principal material direction.

Referring back to equation (53) it is clear that  $C_{11}$  is equal to  $F_1^{-1} * d\Pi_{11}$ , because  $F_1^{-1}$  is the convolution inverse of the creep compliance function it may be considered as the relaxation modulus function. Hence, this tensile stress relaxation test does not directly yield the stress relaxation modulus in the first principal material direction. Provided that other tests are performed such that information on  $C_{22}$ ,  $C_{33}$ , etc. it is possible to determine the specially orthotropic stress relaxation functions. However, the complexity of the expression makes it a practical impossibility to relate the stress measured in this test to a material property function.

## 2.6. Isotropic Viscoelastic Response

Introduction of the isotropic behavior to the analysis of linear viscoelastic response generates significant simplifications in the constitutive relations. For an isotropic material we have the additional matrix symmetry relations:

$$C = C_{ii}, \text{ for } i = 1 \text{ to } 3,$$

$$G = C_{ii}, \text{ for } i = 4 \text{ to } 6, (\text{i.e.: } G = G_{12} = G_{13} = G_{23}),$$

$$L = C_{ij}, \text{ for } i, j = 1 \text{ to } 3, i \neq j$$

$$S = S_{ii}, \text{ for } i = 1 \text{ to } 3, (\text{i.e.: } S = F = F_i, \text{ for } i = 1 \text{ to } 3)$$

$$J = S_{ii}, \text{ for } i = 4 \text{ to } 6, (\text{i.e.: } J = J_{12} = J_{13} = J_{23}), \text{ and}$$

$$M = S_{ij}, \text{ for } i, j = 1 \text{ to } 3, i \neq j,$$

where  $C(t)$  is a normal stiffness relaxation function,  $L(t)$  is a normal stiffness relaxation function due to Poisson's effect,  $M(t)$  is a normal creep compliance function due to Poisson's effect, and  $G(t)$ ,  $J(t)$ , and  $F(t)$  are the isotropic form of the functions previously introduced. With these functions and the above equalities the stiffness matrix, equation (17), becomes:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} = \begin{bmatrix} C & L & L & 0 & 0 & 0 \\ L & C & L & 0 & 0 & 0 \\ L & L & C & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \quad (58)$$

and the compliance matrix, equation (18), becomes:

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} = \begin{bmatrix} F & M & M & 0 & 0 & 0 \\ M & F & M & 0 & 0 & 0 \\ M & M & F & 0 & 0 & 0 \\ 0 & 0 & 0 & J & 0 & 0 \\ 0 & 0 & 0 & 0 & J & 0 \\ 0 & 0 & 0 & 0 & 0 & J \end{bmatrix} \quad (59)$$

Given the isotropic form of the compliance and stiffness matrices, the interrelation of the compliance and stiffness, equation (20), becomes:

$$\begin{bmatrix} C & L & L & 0 & 0 & 0 \\ L & C & L & 0 & 0 & 0 \\ L & L & C & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} * d \begin{bmatrix} F & M & M & 0 & 0 & 0 \\ M & F & M & 0 & 0 & 0 \\ M & M & F & 0 & 0 & 0 \\ 0 & 0 & 0 & J & 0 & 0 \\ 0 & 0 & 0 & 0 & J & 0 \\ 0 & 0 & 0 & 0 & 0 & J \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, (60)$$

which yields three unique equations. They are:

$$C * dF + 2L * dM = 1, \quad (61)$$

$$F * dL + C * dM + L * dM = 0, \quad (62)$$

and

$$G * dJ = 1. \quad (63)$$

Equation (63) is the well known interrelation of the shear creep compliance function and the shear stress relaxation function in the form given by Staverman and Schwarzl. From equations (61) and (62) it is possible to derive expressions for C and L in terms of F and M, and vice versa. These expressions are:

$$C = (F + M) * d \left\{ (F + 2M)^{-1} * d(F - M)^{-1} \right\}, \quad (64)$$

$$L = -M * d \left\{ (F + 2M)^{-1} * d(F - M)^{-1} \right\}. \quad (65)$$

Further, from equations (61) and (62) we find:

$$(C - L) * d(F - M) = 1, \quad (66)$$

$$(C + 2L) * d(F + 2M) = 1, \quad (67)$$

which are easily proved from the results given in equations (64) and (65). The significance of equation (66) is that the inverse convolution relationship between the tensile property functions is held between  $(C - L)$  and  $(F - M)$  as well as between the tensile stress relaxation modulus function, E, and the tensile creep compliance function, F.

Understanding of the above isotropic equations is expanded by the introduction of the viscoelastic Poisson's ratio function. The orthotropic form of the Poisson's ratio function is introduced in equation (37), the isotropic form is simply:

$$\varepsilon_j(t) = - \int_{-\infty}^t \nu(t-\tau) \frac{d\varepsilon_i(\tau)}{d\tau} d\tau, \quad (67)$$

or in convolution notation:

$$\varepsilon_j = -\nu * d\varepsilon_i = -\varepsilon_i * d\nu. \quad (68)$$

As a result of this definition and the results from the orthotropic development the off diagonal terms of the compliance matrix are written as:

$$S_{ij} = M = -\nu * dF, \text{ for } i, j = 1 \text{ to } 3, i \neq j. \quad (69)$$

With this identity and following some manipulation, in which convolution identity (VIII) is of particular importance, equations (64) and (65) become:

$$C = F^{-1} * d \left\{ (1 - \nu) * d \left[ (1 + \nu)^{-1} * d(1 - 2\nu)^{-1} \right] \right\}, \quad (70)$$

and

$$L = F^{-1} * d \left\{ \nu * d \left[ (1 + \nu)^{-1} * d(1 - 2\nu)^{-1} \right] \right\}. \quad (71)$$

Note that if the material is elastic, i.e.  $\nu(t) = \nu_e$ , a constant  $\forall t$ , the above equations reduce to the stiffness values found in the generalized Hooke's law;  $C = \frac{E(1-\nu_e)}{(1-2\nu_e)(1+\nu_e)}$  and  $L = \frac{\nu_e E}{(1-2\nu_e)(1+\nu_e)}$ , where  $E$  is the Young's or elastic modulus. Thus, providing reassurance of the viscoelastic results.

Returning again to the discussion of a uniaxial tensile stress relaxation test, consider the results produced from such a test for an isotropic material. Using the stiffnesses,  $C$  and  $L$ , and an applied strain history:  $\varepsilon_1 = \varepsilon_0 \mathbf{1}$ ,  $\sigma_2 = \sigma_3 = 0$ , ( $\varepsilon_2, \varepsilon_3 \neq 0$ ), we find:

$$\sigma_1 = \left\{ C - 2(L * dL) * d(C + L)^{-1} \right\} \varepsilon_0 \quad (72)$$

and

$$\varepsilon_2 = \varepsilon_3 = -L * d(C + L)^{-1} \varepsilon_0 \quad (73)$$

or in terms of the Poisson's ratio function:

$$\sigma_1 = F^{-1} \varepsilon_0 \quad (74)$$

and

$$\varepsilon_2 = \varepsilon_3 = -\nu \varepsilon_0. \quad (75)$$

The consequence of which is the function found by dividing the measured stress,  $\sigma_1$ , by the applied strain magnitude,  $\varepsilon_0$ , is the tensile stress relaxation modulus of the material. Further, with the introduction of the tensile stress relaxation function,  $E(t)$ , it is clear that  $E(t)$  is equal to  $F^{-1}(t)$ , and therefore:

$$F * dE = 1, \quad (76)$$

which is a classical result and is stated without proof by Staverman and Schwarzl. Using the tensile stress relaxation function,  $E(t)$ , the expressions for the tensile stiffness components,  $C$  and  $L$ , equations (70) and (71), can be rewritten as:

$$C = E * d \left\{ (1 - \nu) * d \left[ (1 + \nu)^{-1} * d(1 - 2\nu)^{-1} \right] \right\}, \quad (77)$$

and

$$L = E * d \left\{ \nu * d \left[ (1 + \nu)^{-1} * d(1 - 2\nu)^{-1} \right] \right\}. \quad (78)$$

These expressions allow direct computation of the two tensile stiffness functions from the stress relaxation function and Poisson's ratio function. It is not apparent from equation (74) or (75) would reduce to a simpler form without the introduction of the Poisson's ratio function.

## **2.7. Poisson's Ratio Function in Isotropic Viscoelastic Materials**

The definition of the Poisson's ratio function given for specially orthotropic material in equation (37) and for an isotropic material in equation (67) do not place any restrictions on the applied normal strain. Staverman and Schwarzl state that there are a number of possible definitions of the Poisson's ratio function for a viscoelastic material. They further state that: "two possibilities are: 1. The ratio of the lateral contraction to longitudinal elongation in a creep experiment. 2. The ratio of lateral contraction to longitudinal elongation in a stress relaxation experiment. For a purely elastic material both definitions are identical. For a viscoelastic material the second possibility has been chosen rather arbitrarily." A proof that in viscoelasticity, as in elasticity, there is only one Poisson's ratio function is presented in this section.

For this proof, the discussion is restricted to isotropic viscoelastic materials. Recalling the definition of the Poisson's ratio function for an isotropic material from equation (67) write:

$$\varepsilon_j(t) = - \int_{-\infty}^t \nu_C(t-\tau) \frac{d\varepsilon_i(\tau)}{d\tau} d\tau \quad (79)$$

and

$$\varepsilon_j(t) = - \int_{-\infty}^t \nu_R(t-\tau) \frac{d\varepsilon_i(\tau)}{d\tau} d\tau, \quad (81)$$

where  $\nu_C(t)$  is the Poisson's ratio function that results from a creep compliance test and  $\nu_R(t)$  is the Poisson's ratio function that results from a stress relaxation test. Again, for convenience use the convolution notation for the above equations, thus:

$$\varepsilon_j = - \nu_C * d\varepsilon_i = - \varepsilon_i * d\nu_C.$$

and

$$\varepsilon_j = - \nu_R * d\varepsilon_i = - \varepsilon_i * d\nu_R.$$

The discussions presented previously for creep compliance testing of a specially orthotropic viscoelastic material and isotropic viscoelastic material are directly applicable to this discussion and the previous conclusions may be used directly with the generalization to isotropic materials. Therefore, using equation (70):

$$S_{ij} = M = - \nu_C * dF, \text{ for } i, j = 1 \text{ to } 3, i \neq j. \quad (82)$$

Define that the off diagonal components of the tensile stiffness matrix, for isotropic materials this is  $L$ , are determined from a stress relaxation test; thus,  $L = L(\nu_R)$ . In the previous development for a uniaxial tension stress relaxation test it was found that:

$$F^{-1} = E = \frac{\sigma_1}{\varepsilon_0}, \quad (83)$$

and

$$\nu_R = L * d(C + L)^{-1}, \quad (84)$$

where  $\nu_R$  is used here to indicate the result is developed from a stress relaxation test. Rearranging equation (84) and solving for  $L$  in terms of  $\nu_R$  and  $C$ , and subsequently



substituting this expression into equation (73) and using (83) results in an expression for C in terms of  $\nu_R$  and E, derived solely from a tensile stress relaxation test. The result is:

$$C = E * d \left\{ (1 - \nu_R) * d \left[ (1 + \nu_R)^{-1} * d(1 - 2\nu_R)^{-1} \right] \right\}. \quad (85)$$

When the above result is substituted back into the intermediate expression for L an expression for L in terms of  $\nu_R$  and E, derived solely from a tensile stress relaxation test, is obtained. This expression is:

$$L = E * d \left\{ \nu_R * d \left[ (1 + \nu_R)^{-1} * d(1 - 2\nu_R)^{-1} \right] \right\}. \quad (86)$$

Equation (66) is a general expression that interrelates the tensile viscoelastic material property functions. It is:

$$(C - L) * d(F - M) = 1. \quad (66)$$

Using equation (82) in rewriting the above relation yields:

$$\left\{ C - L \right\} * d \left\{ F * d \left[ 1 + \nu_C \right] \right\} = 1.$$

The results from the tensile stress relaxation test, equations (85) and (86) are now substituted into the above equation. This yields:

$$\left\{ E * d \left[ \left( 1 + \nu_R \right)^{-1} * d \left( 1 - 2\nu_R \right)^{-1} \right] \right\} * d \left\{ F * d \left[ 1 + \nu_C \right] \right\} = 1.$$

From the interrelation of tensile stress relaxation modulus and tensile creep compliance, equation (77), E and F are eliminated from the above equation. Therefore:

$$\left[ 1 + \nu_C \right] = \left[ 1 + \nu_R \right],$$

which simplifies to:

$$\nu_C = \nu_R. \quad (87)$$

Thus, the Poisson's ratio function found from a creep compliance test in tension,  $\nu_C$ , is the same as the Poisson's ratio function found from a tensile stress relaxation test,  $\nu_R$ . The identical result, equation (87), is obtained from the general expression in equation (67).

To conclude then, there is only one Poisson's ratio function in linear isotropic materials. It may be measured by either a tensile stress relaxation test, or by a tensile creep compliance test.

### 3. An Alternate Form of the Deformation Invariants

The general purpose finite element code **Abaqus** uses an alternate form of the deformation invariants in the constitutive equations implemented in the code. Dr. S. Peng has proposed using a similar alternate formulation for the constitutive equation that he is developing. This section addresses these alternate forms and their relationship to each other and the hyper-elastic constitutive equation.

Define the quantity  $J$  as:

$$J = \det(\mathbf{F}), \quad (1)$$

where  $\mathbf{F}$  is the deformation gradient. Introduce a new measure of the deformation,  $\Phi$ . Where the new measure is defined as:

$$\Phi = J^{-1/3} \mathbf{F}. \quad (2)$$

The Left Cauchy-Green Deformation Tensor,  $\mathbf{B}$ , is defined as:

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T. \quad (3)$$

From this definition a new deformation tensor,  $\mathbf{B}$ , is defined as:

$$\mathbf{B} = \Phi\Phi^T. \quad (4)$$

Therefore:

$$\mathbf{B} = J^{2/3} \mathbf{B}, \quad (5)$$

or

$$\mathbf{B} = J^{-2/3} \mathbf{B}. \quad (6)$$

The Cayley-Hamilton Theorem for the tensor  $\mathbf{B}$  states:

$$\mathbf{B}^3 - I_1 \mathbf{B}^2 + I_2 \mathbf{B} - I_3 \mathbf{I} = \mathbf{0}, \quad (7)$$

where  $I_i$  are defined as the invariants of the tensor  $\mathbf{B}$ . The compatible definition of the invariants is:

$$I_1 = \text{tr}(\mathbf{B}), \quad (8)$$

$$I_2 = \frac{1}{2} \{ I_1^2 - \text{tr}(\mathbf{B}^2) \}, \quad (9)$$

$$I_3 = \det(\mathbf{B}). \quad (10)$$

Note that the new deformation measure,  $\mathbf{B}$ , is a symmetric tensor and therefore must also satisfy the Cayley-Hamilton Theorem. Therefore:

$$\mathbf{B}^3 - \Gamma_1 \mathbf{B}^2 + \Gamma_2 \mathbf{B} - \Gamma_3 \mathbf{I} = \mathbf{0}, \quad (11)$$

where  $\Gamma_i$  are defined as the invariants of the tensor  $\mathbf{B}$ . Similarly, the new invariants are:

$$\Gamma_1 = \text{tr}(\mathbf{B}), \quad (12)$$

$$\Gamma_2 = \frac{1}{2} \left\{ \Gamma_1^2 - \text{tr}(\mathbf{B}^2) \right\}, \quad (13)$$

$$\Gamma_3 = \det(\mathbf{B}). \quad (14)$$

Substitution of the result from equation (6) into equation (12) with equation (8) yields:

$$\Gamma_1 = \text{tr}(J^{2/3} \mathbf{B}) = J^{2/3} \text{tr}(\mathbf{B}) = J^{2/3} \Gamma_1. \quad (15)$$

Similarly, for  $\Gamma_2$ :

$$\Gamma_2 = \frac{1}{2} \left\{ (J^{2/3} \Gamma_1)^2 - \text{tr}((J^{2/3} \mathbf{B})^2) \right\},$$

$$\Gamma_2 = \frac{1}{2} \left\{ J^{4/3} \Gamma_1^2 - J^{4/3} \text{tr}(\mathbf{B}^2) \right\},$$

$$\Gamma_2 = \frac{J^{4/3}}{2} \left\{ \Gamma_1^2 - \text{tr}(\mathbf{B}^2) \right\} = J^{4/3} \Gamma_2, \quad (16)$$

and for  $\Gamma_3$ :

$$\Gamma_3 = \det(J^{2/3} \mathbf{B}) = (J^{2/3})^3 \det(\mathbf{B}) = (J^{2/3})^3 \Gamma_3. \quad (17)$$

Recall equations (1), (3), and (10), they are combined to give:

$$\Gamma_3 = \det(\mathbf{B}) = \det(\mathbf{F}\mathbf{F}^T) = \det(\mathbf{F})\det(\mathbf{F}^T) = \det(\mathbf{F})\det(\mathbf{F})$$

$$\Gamma_3 = J^2. \quad (18)$$

As a result, then:

$$\Gamma_3 = (J^{2/3})^3 J^2 = 1. \quad (19)$$

As a proof that the transformation of the deformation measure is consistent consider equation (7), the Cayley-Hamilton Theorem for the original deformation tensor:

$$\mathbf{B}^3 - \mathbf{I}_1 \mathbf{B}^3 + \mathbf{I}_2 \mathbf{B}^3 - \mathbf{I}_3 \mathbf{I} = \mathbf{0},$$

substitution of the definition of the new deformation measure yields:

$$(J^{2/3} \mathbf{B})^3 - \mathbf{I}_1 (J^{2/3} \mathbf{B})^2 + \mathbf{I}_2 J^{2/3} \mathbf{B} - \mathbf{I}_3 \mathbf{I} = \mathbf{0},$$

$$\mathbf{B}^3 - \mathbf{I}_1 J^{2/3} \mathbf{B}^2 + \mathbf{I}_2 J^{4/3} \mathbf{B} - \mathbf{I}_3 J^{6/3} \mathbf{I} = \mathbf{0}.$$

Then, from equations (15), (16), and (17):

$$\mathbf{B}^3 - \mathbf{I}_1 \mathbf{B}^2 + \mathbf{I}_2 \mathbf{B} - \mathbf{I}_3 \mathbf{I} = \mathbf{0},$$

which is equation (11) the Cayley-Hamilton Theorem for the new deformation measure. Therefore, the definitions of the new deformation measure and the invariants are consistent.

The constitutive equation for a compressible material is given as:

$$\boldsymbol{\sigma} = 2\mathbf{I}_3^{1/2} \left\{ \mathbf{I}_3 \mathbf{W}_3 \mathbf{I} + (\mathbf{W}_1 + \mathbf{I}_1 \mathbf{W}_2) \mathbf{B} - \mathbf{W}_2 \mathbf{B}^2 \right\}, \quad (20)$$

where  $\boldsymbol{\sigma}$  is the stress, and  $\mathbf{W}_i$  is the partial derivative of the strain energy density function,  $W$ , with respect to the  $i$ th invariant of the deformation,  $\mathbf{I}_i$ , i.e.:

$$\mathbf{W}_1 = \frac{\partial W(\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3)}{\partial \mathbf{I}_1}, \quad (21)$$

$$\mathbf{W}_2 = \frac{\partial W(\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3)}{\partial \mathbf{I}_2}, \quad (22)$$

$$\mathbf{W}_3 = \frac{\partial W(\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3)}{\partial \mathbf{I}_3}. \quad (23)$$

Now introduce the new deformation measure into the constitutive equation:

$$\boldsymbol{\sigma} = 2\mathbf{I}_3^{1/2} \left\{ \mathbf{I}_3 \mathbf{W}_3 \mathbf{I} + (\mathbf{W}_1 + \mathbf{I}_1 \mathbf{W}_2) J^{2/3} \mathbf{B} - \mathbf{W}_2 (J^{2/3} \mathbf{B})^2 \right\},$$

$$\boldsymbol{\sigma} = 2\mathbf{I}_3^{1/2} J^{4/3} \left\{ \mathbf{I}_3 \mathbf{W}_3 J^{4/3} \mathbf{I} + (\mathbf{W}_1 + \mathbf{I}_1 \mathbf{W}_2) J^{2/3} \mathbf{B} - \mathbf{W}_2 \mathbf{B}^2 \right\},$$

and with the new invariants:

$$\boldsymbol{\sigma} = 2(J^{6/3} \mathbf{I}_3)^{-1/2} J^{4/3} \left\{ (J^{6/3} \mathbf{I}_3) \mathbf{W}_3 J^{4/3} \mathbf{I} + [\mathbf{W}_1 + (J^{6/3} \mathbf{I}_1) \mathbf{W}_2] J^{2/3} \mathbf{B} - \mathbf{W}_2 \mathbf{B}^2 \right\},$$

$$\boldsymbol{\sigma} = 2\mathbf{I}_3^{1/2} J^{1/3} \left\{ (J^{2/3} \mathbf{I}_3) \mathbf{W}_3 \mathbf{I} + [J^{2/3} \mathbf{W}_1 + (J^{4/3} \mathbf{I}_1) \mathbf{W}_2] \mathbf{B} - \mathbf{W}_2 \mathbf{B}^2 \right\}. \quad (24)$$

Note that  $W_i$  are the partial derivatives of the strain energy density function with respect to the original invariants. To complete the transformation process introduce  $\Omega$ , the strain energy density function expressed as a function of the new variables. Thus:

$$W(I_1, I_2, I_3) = W(J^{2/3}\Gamma_1, J^{4/3}\Gamma_2, J^{6/3}\Gamma_3) = \Omega(\Gamma_1, \Gamma_2, \Gamma_3, J). \quad (25)$$

Therefore, the  $W_i$  become:

$$W_1 = \frac{\partial W(I_1, I_2, I_3)}{\partial I_1} = \frac{\partial \Omega(\Gamma_1, \Gamma_2, \Gamma_3, J)}{\partial \Gamma_1} \frac{\partial \Gamma_1}{\partial I_1} + \frac{\partial \Omega}{\partial J} \frac{\partial J}{\partial I_1},$$

$$W_1 = \frac{\partial \Omega(\Gamma_1, \Gamma_2, \Gamma_3, J)}{\partial \Gamma_1} J^{2/3} = J^{2/3} \Omega_1, \quad (26)$$

$$W_2 = \frac{\partial W(I_1, I_2, I_3)}{\partial I_2} = \frac{\partial \Omega(\Gamma_1, \Gamma_2, \Gamma_3, J)}{\partial \Gamma_2} \frac{\partial \Gamma_2}{\partial I_2} + \frac{\partial \Omega(\Gamma_1, \Gamma_2, \Gamma_3, J)}{\partial J} \frac{\partial J}{\partial I_2},$$

$$W_2 = \frac{\partial \Omega(\Gamma_1, \Gamma_2, \Gamma_3, J)}{\partial \Gamma_2} J^{4/3} = J^{4/3} \Omega_2, \quad (27)$$

$$W_3 = \frac{\partial W(I_1, I_2, I_3)}{\partial I_3} = \frac{\partial \Omega(\Gamma_1, \Gamma_2, \Gamma_3, J)}{\partial \Gamma_1} \frac{\partial \Gamma_1}{\partial I_3} + \frac{\partial \Omega(\Gamma_1, \Gamma_2, \Gamma_3, J)}{\partial \Gamma_2} \frac{\partial \Gamma_2}{\partial I_3} + \frac{\partial \Omega(\Gamma_1, \Gamma_2, \Gamma_3, J)}{\partial \Gamma_3} \frac{\partial \Gamma_3}{\partial I_3} + \frac{\partial \Omega(\Gamma_1, \Gamma_2, \Gamma_3, J)}{\partial J} \frac{\partial J}{\partial I_3},$$

$$W_3 = \frac{\partial \Omega(\Gamma_1, \Gamma_2, \Gamma_3, J)}{\partial J} \frac{J^1}{2} = \frac{J^1}{2} \Omega_3. \quad (28)$$

Substitution of equations (26), (27), and (28) into equation (24) yields:

$$\sigma = 2\Gamma_3^{1/2} J^{1/3} \left\{ (J^{2/3}\Gamma_3) \frac{J^1}{2} \Omega_1 I + [J^{2/3} J^{2/3} \Omega_1 + (J^{4/3}\Gamma_1) J^{4/3} \Omega_2] \mathbf{B} - J^{4/3} \Omega_2 \mathbf{B}^2 \right\},$$

$$\sigma = 2\Gamma_3^{1/2} J^{1/3} \left\{ \frac{J^{1/3}}{2} \Gamma_3 \Omega_1 I + [\Omega_1 + J^{8/3} \Gamma_1 \Omega_2] \mathbf{B} - J^{4/3} \Omega_2 \mathbf{B}^2 \right\},$$

and since  $\Gamma_3 = 1$ ,

$$\sigma = J^{1/3} \left\{ J^{1/3} \Omega_1 I + 2J^{8/3} [J^{8/3} \Omega_1 + \Gamma_1 \Omega_2] \mathbf{B} - 2J^{4/3} \Omega_2 \mathbf{B}^2 \right\},$$

$$\sigma = \Omega_1 I + 2[J^{1/3} \Omega_1 + J^3 \Gamma_1 \Omega_2] \mathbf{B} - 2J^{5/3} \Omega_2 \mathbf{B}^2, \quad (29)$$

with the strain energy density function formulated as:  $\Omega = \Omega(\Gamma_1, \Gamma_2, J)$ . It is not immediately apparent that the expression for the stress in equation (29) is significantly simpler than that given in equation (20). Both forms of the strain energy density functions,  $W$  and  $\Omega$ , are dependent on three variables. Thus, the problem of identifying the strain energy density function is not greatly simplified. Equation (29) does begin to look like the expression for the stress in an incompressible material. In fact for an incompressible material,  $I_3 = 1 = J$ . Then equation (29) becomes:

$$\sigma = \Omega_3 I + 2 \left[ \Omega_1 + \Gamma_1 \Omega_2 \right] B - 2 \Omega_2 B^2, \quad (30)$$

and

$$\sigma = W_3 I + 2 \left[ W_1 + \Gamma_1 W_2 \right] B - 2 W_2 B^2,$$

$$\sigma = W_3 I + 2 \left[ W_1 + I_1 W_2 \right] B - 2 W_2 B^2,$$

$$\sigma = W_3 I + 2 \left[ W_1 + I_1 W_2 \right] B - 2 W_2 B^2.$$

For an incompressible material the constitutive equation is:

$$\sigma = -p I + 2 \left[ W_1 + I_1 W_2 \right] B - 2 W_2 B^2.$$

Therefore, for  $I_3 = J = 1$ ,  $W_3(I_1, I_2, 1) = \Omega_3(\Gamma_1, \Gamma_2, 1) = -p$ . Where  $p$  is an arbitrary hydrostatic pressure that can only be determined by application of the equations of motion and the boundary conditions.

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<sup>1</sup> Staverman, A. J. and Schwarzl, F., "Linear Deformation Behavior of High Polymers," in *Die Physik Der Hochpolymeren*, Vol. 4, ed. by H. A. Stuart, Springer-Verlag, Berlin, 1956.